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Note

Partitions and Sums of (m, p, c)-Sets

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The (m, p, c)-sets characterize those sets which contain solutions to every partition-regular system of homogeneous linear equations. We show that if N is partitioned into finitely many classes there is one class and for each $(m, p, c) \in N \times N \times N$, an (m, p, c) set $B(m, p, c)$ such that all finite sums choosing at most one term from each $B(m, p, c)$ lie in the given class. © 1987 Academic Press, Inc.

Given m, p, and c in N (the set of positive integers) and $\mathbf{x} \in N^m$, we let $S(m, p, c, \mathbf{x}) = \{cx_t + \sum_{i=t+1}^{m} \lambda_i x_i\colon\ t \in \{1, 2, ..., m\}$ and, for $i \in \{t+1, t+2, ..., m\}, \lambda_i \in \{-p, -p+1, ..., p-1, p\}\}$. A set B is called an (m, p, c)-set provided $B \subseteq N$ and there exists $\mathbf{x} \in N^m$ such that $B = S(m, p, c, \mathbf{x})$. The importance of (m, p, c)-sets lies in the fact that they characterize partition regular systems of homogeneous linear equations [1, Satz 2.5] (or see [2, Sect. 3.3]).

It is known (see [1, Satz 3.1 and the following remarks]) that given any partition of N into finitely many classes, some one class contains, for each (m, p, c) in N^3, an (m, p, c)-set. We show in this note that much more is true: given any partition of N into finitely many classes there is one class and, for each (m, p, c) a choice of some (m, p, c)-set, so that all finite sums choosing at most one from each (m, p, c)-set lie in the fixed class.

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We work with the semigroup $(\beta N, +)$, where βN is the Stone–Čech compactification of N and $+$ is the usual left-continuous extension of ordinary addition to βN. The points of βN are ultrafilters. We refer the reader to [3] for an elementary description of $(\beta N, +)$.

DEFINITION. Given (m, p, c) in N^3,

(a) $T(m, p, c) = \{q \in \beta N$: each A in q contains an (m, p, c)-set$\}$.

(b) $U = \bigcap\{T(m, p, c)\colon (m, p, c) \in N^3\}$.

LEMMA 1. *Let $(m, p, c) \in N^3$. Then $T(m, p, c)$ is a (non-empty) compact subsemigroup of $(\beta N, +)$.*

Proof. $T(m, p, c) \neq \varnothing$ by [1, Satz 2.2] and [3, Theorem 6.7]. To see that $T(m, p, c)$ is closed in βN, hence compact, let $q \in \beta N \backslash T(m, p, c)$ and pick A in q such that A contains no (m, p, c)-set. Then $\bar{A} = \{r \in \beta N\colon A \in r\}$ is a neighborhood of q which misses $T(m, p, c)$. To see that $T(m, p, c)$ is a subsemigroup, let q and r be in $T(m, p, c)$. Let $A \in q + r$ and let $B = \{a \in N\colon A - a \in q\}$. Then $B \in r$ and $r \in T(m, p, c)$ so pick $\mathbf{x} \in N^m$ such that $S(m, p, c, \mathbf{x}) \subseteq B$. Let $C = \bigcap\{A - a\colon a \in S(m, p, c, \mathbf{x})\}$. Since $S(m, p, c, \mathbf{x})$ is a finite subset of B, $C \in q$. Pick $\mathbf{y} \in N^m$ such that $S(m, p, c, \mathbf{y}) \subseteq C$ and let $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Then $S(m, p, c, \mathbf{z}) \subseteq A$ as required. ∎

LEMMA 2. *U is a (non-empty) compact subsemigroup of $(\beta N, +)$.*

Proof. Given (m', p', c') and (m'', p'', c'') in N, let $m = \max\{m', m''\}$, $p = \max\{c''p', c'p''\}$, and $c = c'c''$. Then $T(m, p, c) \subseteq T(m', p', c') \cap T(m'', p'', c'')$. Thus, by Lemma 1, U is as required. ∎

We say that a set $R \subseteq N$ is "large for sums of (m, p, c)-sets" provided there exists a choice of an (m, p, c)-set $B(m, p, c)$ for each (m, p, c) in N^3, with $B(m, p, c) \cap B(m', p', c') = \varnothing$ when $(m, p, c) \neq (m', p', c')$, such that $\sum F \in A$ whenever F is a finite non-empty subset of $\bigcup\{B(m, p, c)\colon (m, p, c) \in N^3\}$ with $|F \cap B(m, p, c)| \leqslant 1$ for each (m, p, c) in N^3. The proof of the following theorem uses an old construction due to F. Galvin.

THEOREM. *Whenever N is partitioned into finitely many classes, some one of those classes is large for sums of (m, p, c)-sets.*

Proof. Since U is a compact left-topological semigroup we may pick q in U such that $q + q = q$. (See, e.g., [3, Lemma 8.1].) We show that each member of q is large for sums of (m, p, c)-sets. To this end let $A \in q$.

Enumerate N as $\langle (m, p, c)_n \rangle_{n=1}^{\infty}$. Let $A_1 = A$ and let $C_1 = \{x \in A_1\colon A_1 - x \in q\}$. Since $q + q = q$, $C_1 \in q$. Since $q \in T((m, p, c)_1)$, pick an $(m, p, c)_1$-set, B_1, with $B_1 \subseteq C_1$. Let $n \in N$ and assume we have chosen A_i,

C_i, and B_i for $i<n$. Let $A_n = A_{n-1} \cap \bigcap\{A_{n-1} - x\colon x \in B_{n-1}\}$. Then $A_n \in q$. Let $b = \max \bigcup\{B_i\colon i<n\}$ and let $C_n = \{x \in A_n\colon x > b$ and $A_n - x \in q\}$. Since $q+q=q$ and q is non-principal, $C_n \in q$. Since $q \in T((m, p, c)_n)$, pick an $(m, p, c)_n$-set, B_n, with $B_n \subseteq C_n$.

Now let F be a finite non-empty subset of $\bigcup\{B_n\colon n \in N\}$ with $|F \cap B_n| \leqslant 1$ for each n. We show by induction on $|F|$ that, if $n = \min\{k\colon F \cap B_k \neq \varnothing\}$, then $\sum F \in A_n$. If $F = \{x\}$, then $\sum F = x \in B_n \subseteq C_n \subseteq A_n$. Assume $|F| > 1$, let $n = \min\{k\colon F \cap B_k \neq \varnothing\}$, let $F \cap B_n = \{x\}$, let $G = F \backslash \{x\}$, and let $t = \min\{k\colon G \cap B_k \neq \varnothing\}$. Then $\sum G \in A_t \subseteq A_{n+1} \subseteq A_n - x$ so $\sum F = x + \sum G \in A_n$ as required. ▮

We close by observing that one cannot strengthen the conclusion by allowing sums which choose more than one element from a given (m, p, c)-set. Indeed, let N be partitioned into two cells in such a way that always x and $2x$ are in different cells. Let (m, p, c) be given with $m \geqslant 2$ and $p \geqslant 2$ and let $\mathbf{x} \in N^m$. Then $cx_1 + x_2$, cx_1, and $cx_1 + 2x_2$ are all in $S(m, p, c, \mathbf{x})$ while $cx_1 + (cx_1 + 2x_2) = 2(cx_1 + x_2)$.

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